Matrices and Gaussian Elimination (Part-1)

P. Sam Johnson

National Institute of Technology Karnataka (NITK)
Surathkal, Mangalore, India

Introduction

A **linear equation** is an equation that may be put in the form

$$a_1x_1 + \cdots + a_nx_n = b,$$

where $x_1, \ldots, x_n$ are the variables (or unknowns), and $b, a_1, \ldots, a_n$ are the coefficients, which are often real numbers.

The coefficients may be considered as parameters of the equation, and may be arbitrary expressions, provided they do not contain any of the variables.

A **system of linear equations** (or **linear system**) is a collection of one or more linear equations involving the same variables.

System of linear equations arise in many of the real world problems. The central problem of linear algebra is the solution of linear equations.

We shall discuss few methods of solving systems of linear equations.

An Example of Linear System (Taken from the book "Linear Algebra" by S. Kumaresan)

Example 1.

A shopkeeper offers two standard packets because he is convinced that north indians each more wheat than rice and south indians each more rice than wheat.

- *Packet one* P_1 *: 5kg wheat and 2kg rice ;*
- *Packet two* P_2 *: 2kg wheat and 5kg rice.*

Notation. (m, n) *: m kg wheat and n kg rice.*
Suppose I need 19kg of wheat and 16kg of rice. Then I need to buy x *packets of* P_1 *and* y *packets of* P_2 *so that* $x(5, 2) + y(2, 5) = (10, 16)$*.*
Hence we need to find x *and* y *such that each of the following equations is satisfied :*

$$5x + 2y = 10$$
$$2x + 5y = 16.$$

Another Example

Example 2.

A bacteriologist has placed three types of bacteria, labelled B_1, B_2 and B_3, in a culture dish, along with certain quantities of two nutrients, labeled, N_1 and N_2. The amounts of each nutrient that can be consumed by each bacterium in a 24-hour period is given below.

	B_1	B_2	B_3
N_1	1	2	6
N_2	3	0	2

We now formulate a mathematical problem to find how many bacteria of each type can be supported daily by 4000 units of N_1 and 1200 units of N_2. Let x, y and z be the number of bacteria of each type represented in the culture. Then the problem is to find the values x, y and z such that

$$x + 2y + 6z = 4000$$
$$3x + 2z = 1200.$$

Geometry of Linear Equations : Row Picture

Suppose we have n equations with n unknowns.

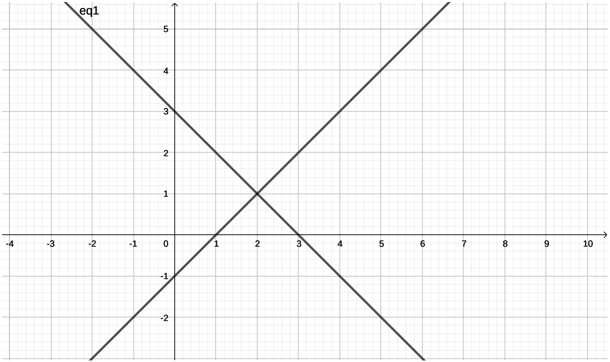
Each equation represents a $(n-1)$-dimensional plane in n-dimensional space. The first two equations intersect (we hope) in a smaller set of "dimension $n-2$".

Assuming all goes well, every new plane (every new equation) reduces the dimension by one. At the end, when all n planes are accounted for, the intersection has dimension zero.

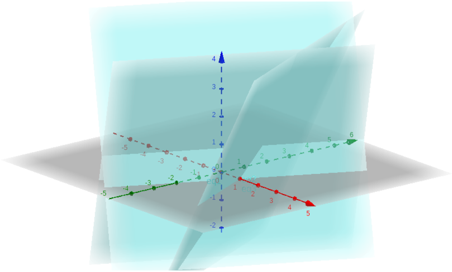
It is a point, it lies on all the planes, and its coordinates satisfy all n equations. It is the solution.

Row picture is a graphical picture : With n equations in n unknowns, there are n-planes in the row picture. Intersection (solution) of n planes (each plane is of $n-1$ dimension).

Consider $\begin{cases} x + y = 3 \\ x - y = 1 \end{cases}$



$$\begin{cases} -2x + y + z = 0 \\ 2x - y + z = 2 \\ 2x + y - z = 2 \end{cases}$$



Row Picture : Singular Case

The geometry exactly breaks down, in what is called the "**singular case**". For instance, **sigular case in** 2 **equations with** 2 **unknowns :**

$$\begin{cases} x + \ \ y = 3 \\ 2x + 2y = 10 \end{cases}$$ Two lines are parallel - no solution.

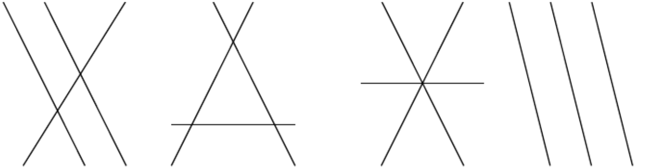
$$\begin{cases} x + \ \ y = 3 \\ 2x + 2y = 6 \end{cases}$$ Two lines are the same - infinitely many solutions.

Sigular case in 3 **equations with** 3 **unknowns :**

- All planes are parallel - no solution ;
- Two planes are parallel - no solution ;
- No common intersection - no solution ;
- Intersection of three planes is a line - infintely many solutions.

Row Picture : Singular Case

Look at the following figure assuming all planes are perpendicular to the screen.[1]



Singular cases: no solution for (a), (b), or (d), an infinity of solutions for (c).

[1] The picture is taken from "Linear Algebra and Its Applications" by Gilbert Strang.

Column Picture (Algebraic Picture)

We say that b is a **linear combination** of the vectors $v_1, v_2, \ldots, v_n$ if

$$b = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n$$

for some real numbers $x_1, x_2, \ldots, x_n$.

With n equations in n unknowns, there are n vectors in the column picture, plus a vector b on the right side. The right side b is a linear combination of the column vectors. Solution is the coefficients in the linear combination of columns.

The n separate equations are really one "vector equation".

$$x_1 [\vdots] + \cdots + x_n [\vdots] = b.$$

The problem is to find the combination of the column vectors on the left side which produces the vector on the right side.

Column Picture

Suppose "three column vectors" span a plane. Suppose if the vector b is not in that plane, then "no solution" case.

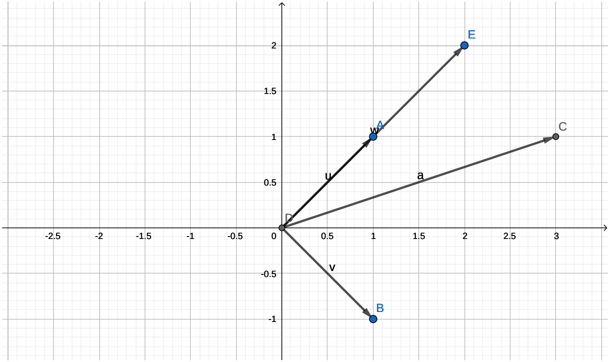
Suppose b lies in the plane of the columns, there are too many solutions.

In that case the **three columns can be combined in infinitely many ways to produce b. How do we know that the three columns lie in the same plane? We will check whether the three column vectors are linearly independent or not?**

We shall discuss linearly independent sets later.

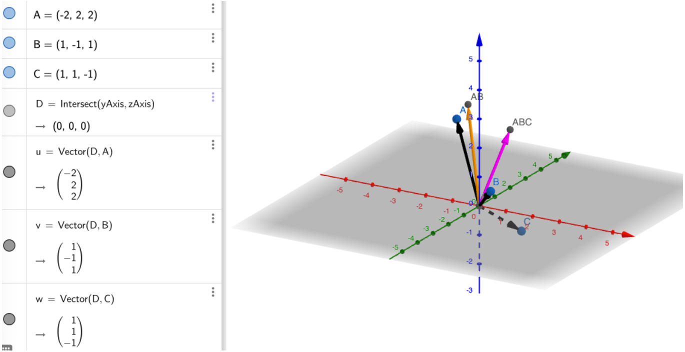
Geometry of Linear Equations : Column Picture

Rewrite the linear system as : $x \begin{pmatrix} 1 \\ 1 \end{pmatrix} + y \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$



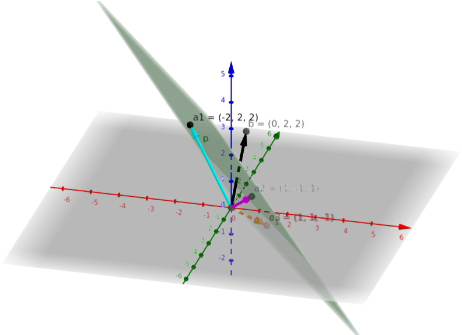
Geometry of Linear Equations : Column Picture

$$x \begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix} + y \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + z \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} \implies xa_1 + ya_2 + za_3 = b$$



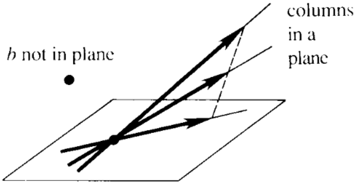
Column Picture : Singular Case

$$\begin{cases} x + y + \ \ z = 2 \\ 2x \qquad + 3z = 5 \\ 3x + y + 4z = 6 \end{cases} \implies x\begin{pmatrix}1\\2\\3\end{pmatrix} + y\begin{pmatrix}1\\0\\1\end{pmatrix} + z\begin{pmatrix}1\\3\\4\end{pmatrix} = \begin{pmatrix}2\\5\\6\end{pmatrix} \implies xa_1 + ya_2 + za_3 = b$$

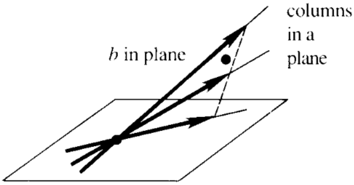


The vector $b = \begin{pmatrix}2\\5\\6\end{pmatrix}$ lies on the plane parallel to the one generated by above three column vectors. Hence there is no solution.

Column Picture : Singular Case



(a) no solution



(b) infinity of solutions

Summary

We consider n linear equations in n unknowns.

1. Row picture : Intersection of $(n-1)$-dimensional planes.
2. Column picture : Linear combination of columns.
3. Linear combination involves "vector addition" and "scalar multiplication (multiply a vector by a scalar)"

Exercises 3.

1. *Write examples of system of equations which do not have any solution in*
   - (a) *one variable, one equation*
   - (b) *one variable, two equations*
   - (c) *two variables, one equation*
   - (d) *two variables, two equations*
   - (e) *three variables, two equations*
   - (f) *three variables, three equations*
2. *Sketch these three lines and decide if the equations are solvable:*

$$\text{3 by 2 system} \qquad \begin{aligned} x + 2y &= 2 \\ x - y &= 2 \\ y &= 1. \end{aligned}$$

What happens if all right-hand sides are zero? Is there any nonzero choice of right-hand side that allows the three lines to intersect at the same point?

Exercises 4.

1. *Describe the intersection of the three planes* $u + v + w + z = 6$ *and* $u + w + z = 4$ *and* $u + w = 2$ *(all in four-dimensional space). Is it a line or a point or an empty set? What is the intersection if the fourth plane* $u = -1$ *is included? Find a fourth equation that leaves us with no solution.*
2. *Give two more right-hand sides in addition to* $b = (2, 5, 7)$ *for which equation (4) can be solved. Give two more right-hand sides in addition to* $b = (2, 5, 6)$ *for which it cannot be solved.*
3. *Draw the two pictures in two planes for the equations* $x - 2y = 0$, $x + y = 6$.
4. *When equation 1 is added to equation 2, which of these are changed: the planes in the row picture, the column picture, the coefficient matrix, the solution?*
5. *If* (a, b) *is a multiple of* (c, d) *with* $abcd \neq 0$*, show that* (a, c) *is a multiple of* (b, d)*. This is surprisingly important: call it a challenge equation. You could use numbers first to see how* $a, b, c,$ *and* d *are related. The question will lead to :*

$$\textit{If } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \textit{ has dependent rows, then it has dependent columns.}$$

Exercises 5.

1. *The first of these equations plus the second equals the third:*

$$\begin{aligned} x + y + z &= 2 \\ x + 2y + z &= 3 \\ 2x + 3y + 2z &= 5. \end{aligned}$$

   The first two planes meet along a line. The third plane contains that line, because if x, y, z *satisfy the first two equations then they also* _______. *The equations have infinitely many solutions (the whole line* L*). Find three solutions.*

2. *Normally* 4 *"planes" in four-dimensional space meet at a* _______. *Normally 4 column vectors in four-dimensional space can combine to produce* b. *what combination of* $(1,0,0,0)$, $(1,1,0,0)$, $(1,1,1,0)$, $(1,1,1,1)$ *produces* $b = (3,3,3,2)$? *What 4 equations for* x, y, z, t *are you solving?*

How to solve a system of linear equations?

Reducing the number of equations :

$$\begin{cases} x + \ \ y + \ \ z = 1 \\ x - 2y + \ \ z = 2 \\ x + \ \ y - 2z = 0 \end{cases} \implies \begin{cases} 3x + 3z = 4 \\ 3x - 3z = 2 \end{cases} \implies \{6x = 6 \, .$$

There are many ways to reduce the number of equations.

Exercise 6.

Find x, then evaluate z and obtain y from one of the original equations.

What happens, if the system is of size 100×100?

Johann Carl Friedrich Gauss

A systematic way of doing this elimination process is called *Gaussian Elimination* by "Johann Carl Friedrich Gauss (1810)".

Johann Carl Friedrich Gauss (30 April 1777 - 23 February 1855) was a German mathematician, geodesist, and physicist who made significant contributions to many fields in mathematics and science. Gauss ranks among history's most influential mathematicians[2].

[2] Source : Wikipedia

Gaussian Elimination - An Example

The way to understand the procedure of Gaussian elimination is by example. We begin in three dimensions with the system

$$\begin{aligned} 2u + v + w &= 5 \\ 4u - 6v &= -2 \\ -2u + 7v + 2w &= 9. \end{aligned}$$

The method starts by substracting multiples of the first equation from the others, so as to eliminate u from the last two equations. This requires that we

(a) subtract 2 times the first equation from the second;

(b) substract -1 times the first equation from the third.

Gaussian Elimination - An Example

The result is an equivalent system of equations

$$\begin{aligned} 2u + v + w &= 5 \\ -8v \quad -2w &= -12 \\ 8v + 3w &= 14. \end{aligned}$$

The coefficient 2, which multiplied the first unknown u in the first equation, is known as the **first pivot**.

Elimination is constantly dividing the pivot into the numbers underneath it, to find out the right multipliers.

At the second stage of elimination, we ignore the first equation. We add the second equation to the third or, in other words, we "substract -1 times the second equation from the third".

Gaussian Elimination - An Example

The elimination process is now complete, at least in the "forward" direction.

$$\begin{aligned} 2u + v + w &= 5 \\ -8v \quad -2w &= -12 \\ w &= 2. \end{aligned}$$

There is an obvious order in which to solve this system. The last equation gives $w = 2$. Substituting into the second equation, we find $v = 1$. Then the first equation gives $u = 1$. The process is called **back-substitution**. Forward elimination produced the pivots $2, -8, 1$. It subtracted multiples of each row from the rows beneath. It reached the "triangular" system. Then this system was solved in reverse order, from bottom to top, by substituting each newly computed value into the equation above. By definition, **pivots cannot be zero**. We need to divide by them.

Summary

$$\begin{cases} 2u + v + w = 5 \\ 4u - 6v = -2 \\ -2u + 7v + 2w = 9 \end{cases}$$

$Eqn\ 2 \rightarrow Eqn\ 2\ -\ (2) \times\ Eqn1$
$Eqn\ 3 \rightarrow Eqn\ 3\ -\ (-1) \times\ Eqn1$

$$\begin{cases} 2u + v + w = 5 \\ -8v - 2w = -12 \\ 8v + 3w = 14 \end{cases}$$

$Eqn\ 3 \rightarrow Eqn\ 3\ -\ (-1) \times\ Eqn2$

$$\begin{cases} 2u + v + w = 5 \\ -8v - 2w = -12 \\ (1)w = 2 \end{cases}$$

- The numbers 2, -8 and 1 are known as pivot elements of the elimination (First leading entries in row 1, 2 and 3 in the last system after elimination).
- These pivot elements and the coefficients of the corresponding variables in subsequent equations decides the multipliers 2, -1 and -1 used in above elimination. How?
- From the last system (after completing elimination): $w = 2 \implies v = (-12 + 2 \times w)/(-8) = 1 \implies u = (5 - w - v)/2 = 1$
- Two processes: Forward elimination and Backward substitution.

Gaussian Elimination - An Example

We include the vector b as the last column :

$$\begin{pmatrix} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{pmatrix} \xrightarrow{R_2 \to R_2 - 2 \times R_1, \quad R_3 \to R_3 - (-1) \times R_1}$$

$$\begin{pmatrix} 2 & 1 & 1 & 5 \\ & -8 & -2 & -12 \\ & 8 & 3 & 14 \end{pmatrix} \xrightarrow{R_3 \to R_3 - (-1) \times R_2}$$

$$\begin{pmatrix} 2 & 1 & 1 & 5 \\ & -8 & -2 & -12 \\ & & 1 & 2 \end{pmatrix}$$

Gaussian Elimination - Algorithm

Imagine a 100×100 linear system to be solved. Then you need to do the elimination very systematically so that you/your computer remember the steps as much as possible. Basic steps involved are:

1. Eliminate the first variable from all equations, except equation 1 .
2. Each row, we identify a nonzero number (which is called a pivot) as far as possible.
3. It is needed to remember the operations performed during elimination.
4. Do similarly for other columns and other variables.
5. Obtain an upper triangular system which can be solvable easily by back substitution.

Linear System into Matrix Form

The matrix notation araises naturally from the system of linear equations.

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1\\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{1n}x_n &= b_2\\
\vdots \qquad\qquad\qquad & \quad \vdots\\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

can be written as

$$Ax = b$$

where $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$.

Coefficient and Augmented Matrices

The matrix $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}$ is called the **coefficient matrix**.

The matrix $[A \mid b] = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{pmatrix}$ is called the **augmented matrix**.

Notations

- $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}, i = 1, 2, \ldots, n\}$.
- We denote the set of all matrices of order $m \times n$ by $\mathbb{R}^{m \times n}$.
- An element $(x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$ is also denoted either as a **column matrix** $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ or a **row matrix** $\begin{pmatrix} x_1 & x_2 & \ldots & x_n \end{pmatrix}$.

Let us consider the linear system $Ax = b$.

STEP 1 Start with the augmented matrix $[A \mid b]$.

STEP 2 We assume that $a_{11} \neq 0$. Call it the first pivot. If $a_{11} = 0$, we require a row exchange which will be discussed later. Row exchanges will be discussed later when the system is not singular. Then the exchanges produce a full set of pivots. For the present we trust all n pivot entries to be nonzero, without changing the order of the equations. That is the best case, with which we continue. Evaluate the multipliers $\ell_{j1} = \frac{a_{j1}}{a_{11}}, \quad j = 2, \ldots, n$. That is, a_{11} is the pivot for the first row. Calculate new rows using $a'_{jk} = a_{jk} - \ell_{j1} a_{1k}, \quad k = 1, \ldots, n+1$. This leads to

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \\ a_{n1} & a_{n2} & \ldots & a_{nn} & b_n \end{pmatrix} \Longrightarrow \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ 0 & a'_{22} & \ldots & a'_{2n} & b'_2 \\ 0 & a'_{32} & \ldots & b'_{3n} & b'_3 \\ \vdots & \vdots & \ddots & \vdots & . \\ 0 & a'_{n2} & \ldots & a'_{nn} & b'_n. \end{pmatrix}$$

Gaussian Elimination - Algorithm

STEP 3 If $a'_{22} \neq 0$, then pivot element for Row 2 is a'_{22} with multipliers $\ell_{j2} = \frac{a'_{j2}}{a'_{22}}, \quad j = 3, \ldots, n.$ Elimination is done on $(n-1) \times n$ matrix,

$$\begin{pmatrix} a'_{22} & \ldots & a'_{2n} & b'_2 \\ a'_{32} & \ldots & a'_{3n} & b'_3 \\ \vdots & & \ddots & \vdots \\ a'_{n2} & \ldots & a'_{nn} & b'_n \end{pmatrix}.$$

That is, $a'_{jk} = a_{jk} - \ell_{j2} a_{2k}, \; k = 2, \ldots, n+1.$

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ 0 & a'_{22} & \ldots & a'_{2n} & b'_2 \\ 0 & a'_{32} & \ldots & a'_{3n} & b'_3 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & a'_{n2} & \ldots & a'_{nn} & b'_n \end{pmatrix} \Longrightarrow \begin{pmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} & b_1 \\ 0 & a'_{22} & a'_{23} & \ldots & a'_{2n} & b'_2 \\ 0 & 0 & a''_{33} & \ldots & a''_{3n} & b''_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & a''_{n3} & \ldots & a''_{nn} & b''_n \end{pmatrix}$$

STEP 4 STOP elimination process once we reached the last row.

The breakdown of elimination.

Under what cicumstances could the process break down?

Something must go wrong in the singular case, and something might go wrong in the nonsingular case. The question is not geometric but algebraic. If the algorithm produces n pivots, thenthere is only one solution to the equations.

The system is nonsingular, and it is solved by forward elimination and back-substitution. But if a zero appears in a pivot position, elimination has to stop - either temporarily or permanently. The sytem might or might not be singular.

Notice that a zero can appear in a pivot position, even if the original coefficient in that place was not zero. Roughly speaking, we do not know whether a zer will appear until we try, by actually going through the elimination process.

Nonsingular case example (cured by exchanging equations)

Example 7.

$$\begin{cases} 3x + 4y + 7z = 6 \\ 6x + 8y + 3z = 7 \\ \ \ x + 2y + \ \ z = 2 \end{cases} \qquad \textit{pivot} = 3, \quad \textit{multipliers } \ell_{21} = 2, \ell_{31} = \frac{1}{3}$$

$$\begin{cases} 3x + 4y + \ \ 7z = 6 \\ \qquad\quad\ - 11z = -5 \\ \qquad \frac{2}{3}y - \ \frac{4}{3}z = 0 \end{cases}$$

Coefficient of y in the second equation is zero. There is a temporary failure *which can be corrected by row exchanges.*

Row 2 $\leftrightarrow$ *Row* 3

$$\begin{cases} 3x + 4y + \ \ 7z = 6 \\ \qquad \frac{2}{3}y - \ \frac{4}{3}z = 0 \\ \qquad\quad\ - 11z = -5 \end{cases} \qquad \textit{Backward substitution gives } z = \frac{5}{11}, y = \frac{10}{11}, x = \frac{-3}{11}.$$

Example - Gaussian Elimination

Example 8.

$$\begin{cases} 3x + 4y + 7z = 6 \\ 5x + 8y + 9z = 10 \\ x + 2y + z = 2 \end{cases} \qquad \textit{pivot} = 3, \textit{ multipliers } \ell_{21} = \frac{5}{3}, \ell_{31} = \frac{1}{3}$$

$$\begin{cases} 3x + 4y + 7z = 6 \\ \frac{4}{3}y - \frac{8}{3}z = 0 \\ \frac{2}{3}y - \frac{4}{3}z = 0 \end{cases} \qquad \textit{pivot} = \frac{4}{3}, \textit{ multiplier } \ell_{32} = \frac{1}{2}$$

$$\begin{cases} 3x + 4y + 7z = 6 \\ \frac{4}{3}y - \frac{8}{3}z = 0 \end{cases}$$

We have two equations. They cannot be simplified further through elimination.

In this example, infinitely many solutions exist.

Summary

1. Gaussian elimination (including forward elimination and backward substitution) helps to find the unique solution of $Ax = b$, if it exists.
2. We have a singular case : **Gaussian elimination breaks down temporarily** in the following situation.
   If a pivot element is zero, include row exchange at every step in above Gaussian algorithm to ensure pivot elements are non-zero.
3. We have a singular case : **Gaussian elimination breaks down permanently** in the following two situations.
   - System has "infinitely many solutions", if there are no pivot elements (pivot element is a non-zero element) in the last column of the transformed (with row operations) augmented matrix $[A \mid b]$.
   - System has "no solution", if last column of the transformed $[A \mid b]$ has a pivot element.

Elemetary Row Operations

1. Add a non-zero scalar multiple of one row to another :

$$i^{th} Row \rightarrow i^{th} Row + \ell_{ji}(j^{th} Row), \quad \text{for } i \neq j.$$

2. Multiply a row by a non-zero scalar factor :

$$i^{th} Row \rightarrow c(i^{th} Row), \quad \text{for } c \neq 0.$$

3. Interchange a pair of rows :

$$i^{th} Row \leftrightarrow j^{th} Row.$$

- **Two matrices are row equivalent to each other**, if each can be obtained from the other by applying a sequence of permitted row operations.
- Let two linear systems be represented by their augmented matrices. **If these two augmented matrices are row equivalent to each other, then the solutions of the two systems are identical.**

For a rectangular system

Gaussian Elimination is not useful in solving the system, but it is helpful in identifying consistency of the system.

Example 9.

$$\begin{pmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 9 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 3 & -7 & 8 & -5 & 9 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{pmatrix}$$

$$\xrightarrow{R_3 \leftrightarrow R_3 - R_1} \begin{pmatrix} 3 & -7 & 8 & -5 & 9 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \\ 0 & -2 & 4 & -4 & -2 & 6 \end{pmatrix}$$

$$\xrightarrow{R_3 \to R_3 + \frac{2}{3} R_2} \begin{pmatrix} 3 & -7 & 8 & -5 & 9 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \\ 0 & 0 & 0 & 0 & \frac{2}{3} & \frac{8}{3} \end{pmatrix}$$

The system is consistent but has infinitely many solutions.

Example - Gaussian Elimination

Let m and n be number of rows and columns of a matrix A (m equations, n unknowns). Let r be the number of pivot elements (non-zero leading entry in each row) identified.

1. If $r = m$ and $m = n$ (full set of pivots), then the system has a unique solution for any vector b.
2. If $r = m$ and $m < n$, then $n - r = n - m$ variables are free. There are infinitely many solutions.
3. If $r < m$, then the last $n - r$ rows of A (transformed) become zero.
   - If **number of zero rows** of $A =$ number of zero rows of $[A \mid b] = n - r$, then system is consistent.
   - If **number of zero rows** of $A = n - r >$ number of zero rows of $[A \mid b]$, then system is inconsistent.

What about the case $r = m$ and $m > n$?

Cost of Gaussian Elimination

For n equations in n unknowns, how many separate arithmetical operations does elimination require?

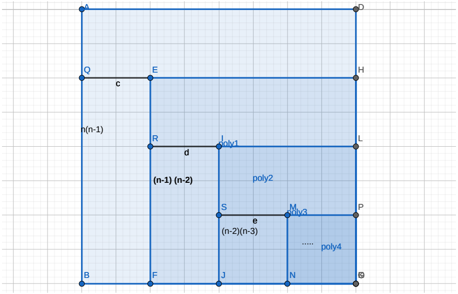


Figure: For an $n \times n$ system

Cost of Gaussian Elimination

For the moment, we ignore the right-hand sides of the equations, and count only operations on the left.

These operations are of two kinds.

1. One is a **division** by the pivot, to find out what multiple (say ℓ) of the pivot equation is to be subtracted.
2. Second is **multiplication-subtraction** ; the terms in the pivot equation are multiplied by ℓ, and then subtracted from the equation beneath it.

Cost of Gaussian Elimination

Suppose we call each division, and each multiplication-subtraction, a single operation. There are $n-1$ rows underneath the first one, so the first stage of elimination needs $n(n-1) = n^2 - n$ operations. (Another approach to $n^2 - n$ is this: All n^2 entries need to be changed, except the n in the first row). When the elimination is down to k equations, only $k^2 - k$ operations are needed to clear out the column below the pivot- by the same reasoning that applied to the first stage, when k equaled n. Altogether, the total number of operations on the left side of the equations is $\sum_{k=1}^{n} k(k-1) = (n^3 - n)/3$. Forward elimination is about a thrid of a million steps, a good code on a PC would take 41 seconds. If n is at all large, a good estimate for the number of operators is $n^3/3$.

Back substitution is considerably faster. The last unknown is found in only one operation (a division by the last pivot). The second to last unknown requires two operations, and so on. Then the total for back-substitution is $\sum_{k=1}^{n} = n(n+1)/2 \approx n^2/2$.

Cost of Gaussian Elimination

From A to upper triangular U :

Stage	Addns/Subns	Multiplications	Divisions
1	$(n-1)^2$	$(n-1)^2$	$n-1$
2	$(n-2)^2$	$(n-2)^2$	$n-2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-1$	1	1	1
	$\frac{(n-1)n(2n-1)}{6}$	$\frac{(n-1)n(2n-1)}{6}$	$\frac{(n-1)n}{2}$

Elimination on b :

Stage	Addns/Subns	Multiplications
1	$(n-1)$	$(n-1)$
2	$(n-2)$	$(n-2)$
$\vdots$	$\vdots$	$\vdots$
$n-1$	1	1
	$\frac{(n-1)n}{2}$	$\frac{(n-1)n}{2}$

Cost of Gaussian Elimination

Back Substitution :

Stage	Addns/Subns	Multiplications	Divisions
x_n	0	0	1
x_{n-1}	1	1	1
x_{n-2}	2	2	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_1	$(n-1)$	$(n-1)$	1
	$\frac{(n-1)n}{2}$	$\frac{(n-1)n}{2}$	n

1. **Total number of operations:** $\frac{2}{3}n^3 + \frac{3}{2}n^2 - \frac{7}{6}n$ (verify!)
2. For n large, the total cost can be considered in the order of n^3. Look at the power of n.
3. Are there different methods so that the order be less than n^3? Yes. Actually it is $Cn^{\log_2 7}$ while $Cn^{\log_2 8} = Cn^3$.

Exercises 10.

1. *Solve using Gauss Elimination:* $\begin{cases} 2u + 3v \qquad = 0 \\ 4u + 5v + \ \ w = 3 \\ 2u - \ \ v - 3w = 5 \end{cases}$.
2. *Find three values of a for which elimination breaks down in*

$$au + v = 1; \ 4u + av = 2$$

3. *Solve the system and find the pivots when* $\begin{cases} 2u - \ \ v \qquad\qquad = 0 \\ -u + 2v - \ \ w \qquad = 0 \\ \quad - \ \ v + 2w - \ \ z = 0 \\ \qquad\quad - \ \ w + 2z = 5. \end{cases}$

   This kind of system has a name. Find out. Also Gauss Elimination gets lot simplified for such systems! Find the name of the algorithm.

Exercises 11.

1. *Using Gauss Elimination, solve* $\begin{cases} u+\ \ v+\ \ w=n \\ u+2v+2w=n^2 \\ 2u+3v-4w=n^3 \end{cases}$, *for each* $n \in \mathbb{N}$.

2. *Find* α, β, γ *and* δ *such that*

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix}, \text{ for all } a, b, c \text{ and } d.$$

3. *Find* α, β, γ *and* δ *such that*

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & d - \frac{c}{a} b \end{pmatrix}, \text{ for all } a, b, c \text{ and } d.$$

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